Virtual Knots, The Arrow Polynomial, and Checkerboard Colorability

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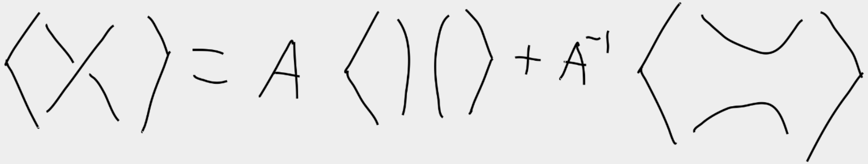
Recalling the Bracket Polynomial

$$\langle \bigcirc \rangle = 1$$

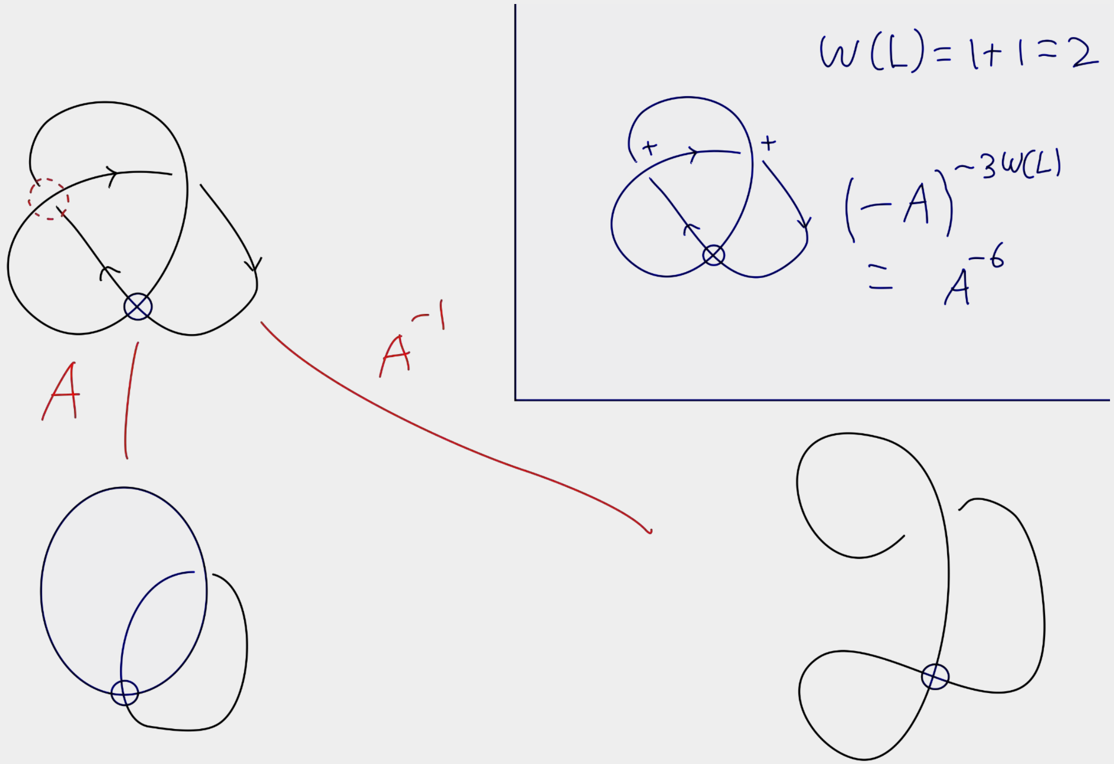


$$\langle \bigcirc \sqcup L \rangle = d \langle L \rangle$$

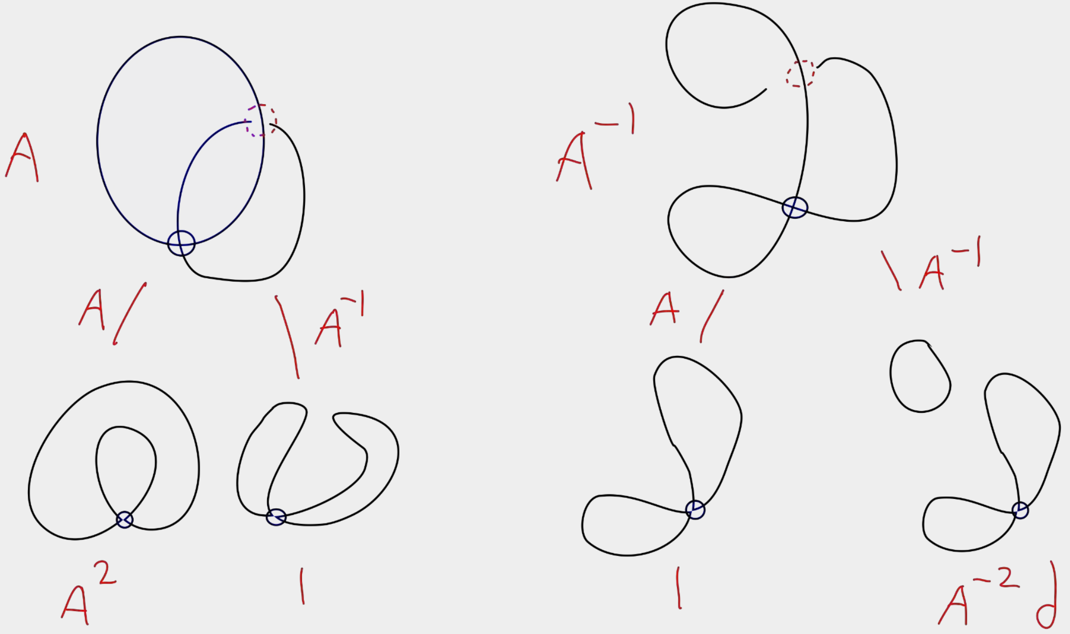
$$X(L) = (-A^{3})^{-w(L)} \langle L \rangle$$

$A = t^{-1/4} \rightarrow$ Jones $\qquad d = (-A^{2} - A^{-2})$

Recalling the Bracket Polynomial

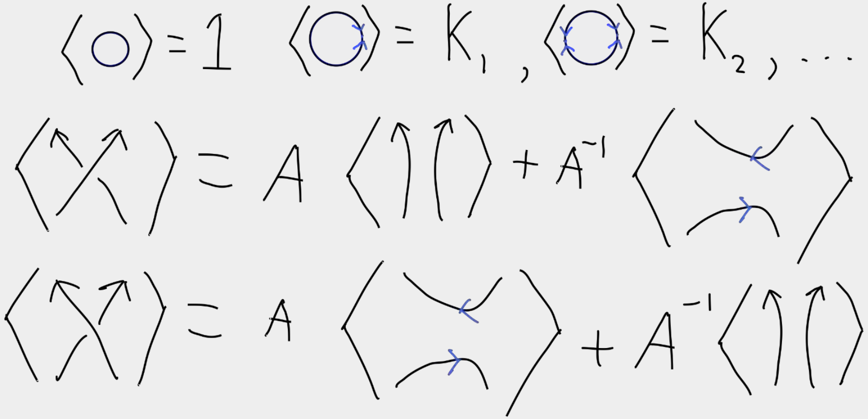


Recalling the Bracket Polynomial



$$X(L) = A^{-6}\left(A^{2} + 2 + A^{-2}d\right)$$

The arrow polynomial

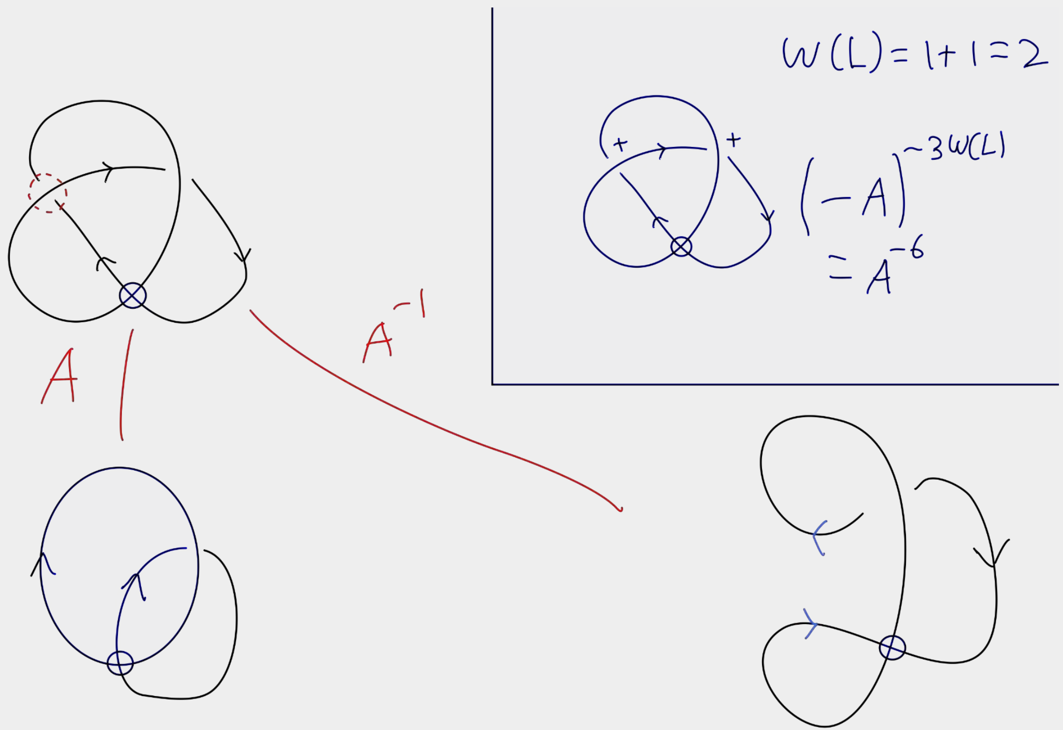


$$\langle O \sqcup L \rangle = d \langle L \rangle, \quad \langle K_i \sqcup L \rangle = K_i \langle L \rangle$$

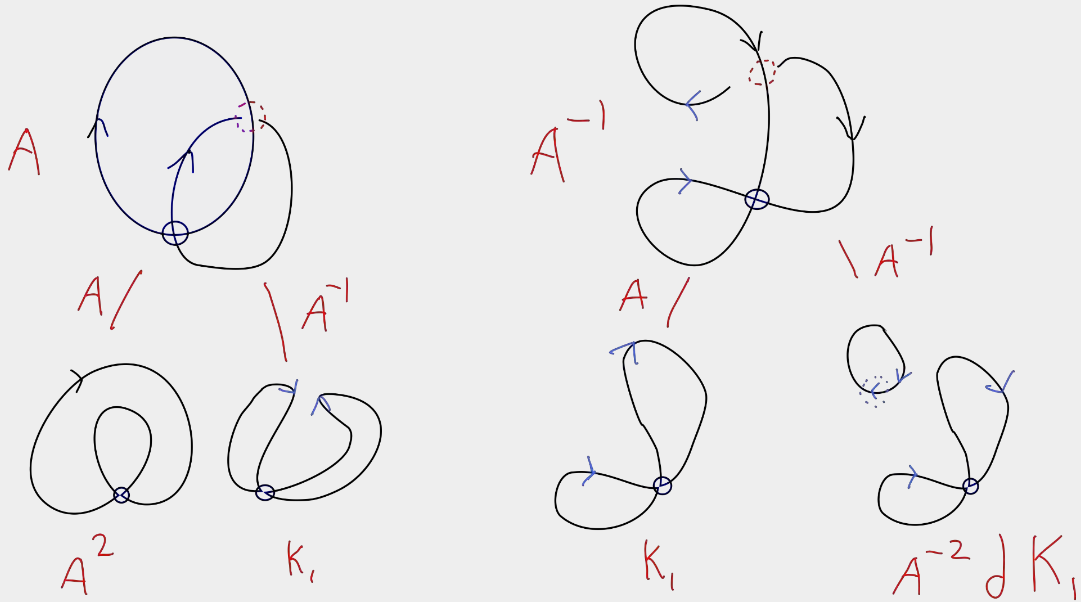
$$NA(L) = (-A^3)^{-w(L)} \langle L \rangle$$

$$d = (-A^2 - A^{-2})$$

The arrow polynomial



The arrow polynomial



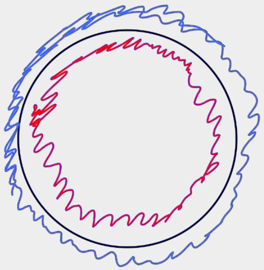
$$NA(L) = A^{-6}\left(A^{2} + (2 + A^{-2}d)K_1\right)$$

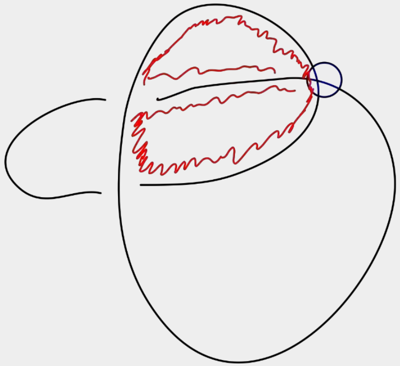
Relation to Virtual Crossing number

Theorem 2.3. *Let K be a virtual link diagram. Then the virtual crossing number of K, $v(K)$, is greater than or equal to the maximum k-degree of $\langle K \rangle_A$.*

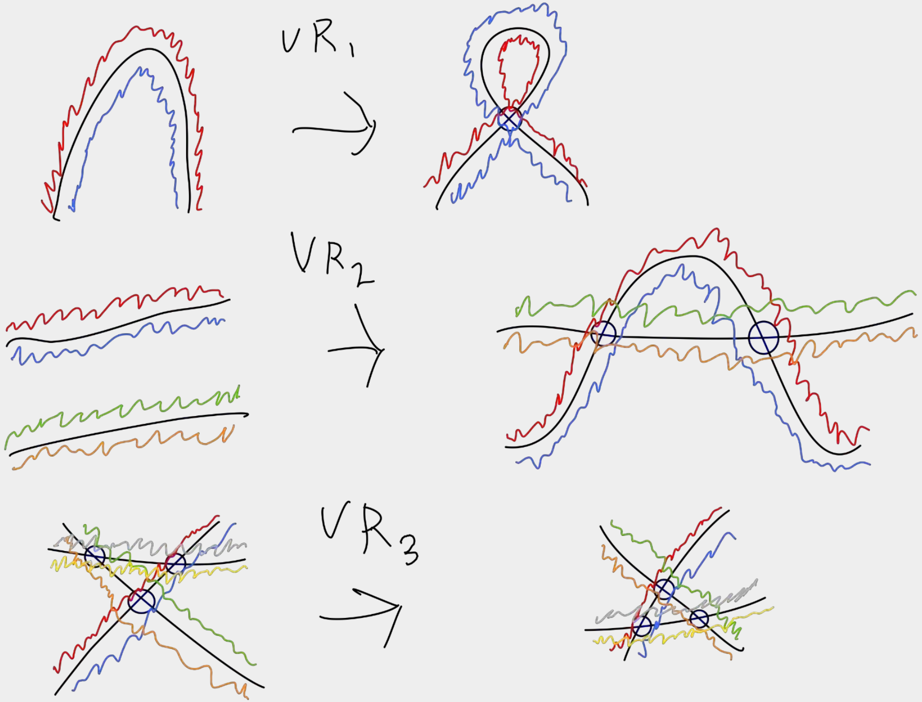
Dye-Kauffman (2018).

Checkerboard Colorability

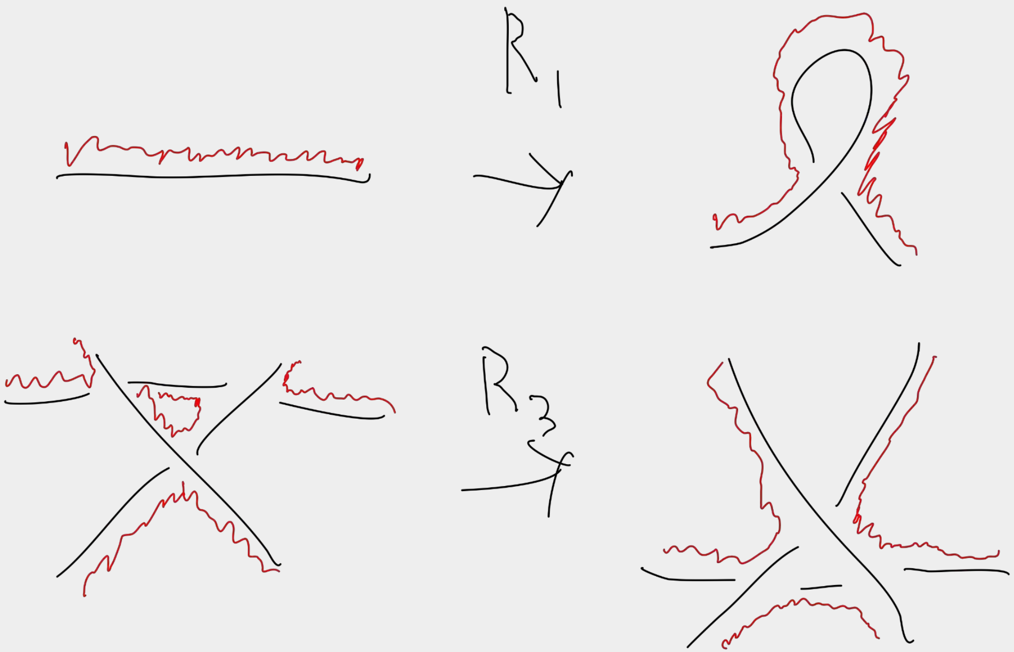




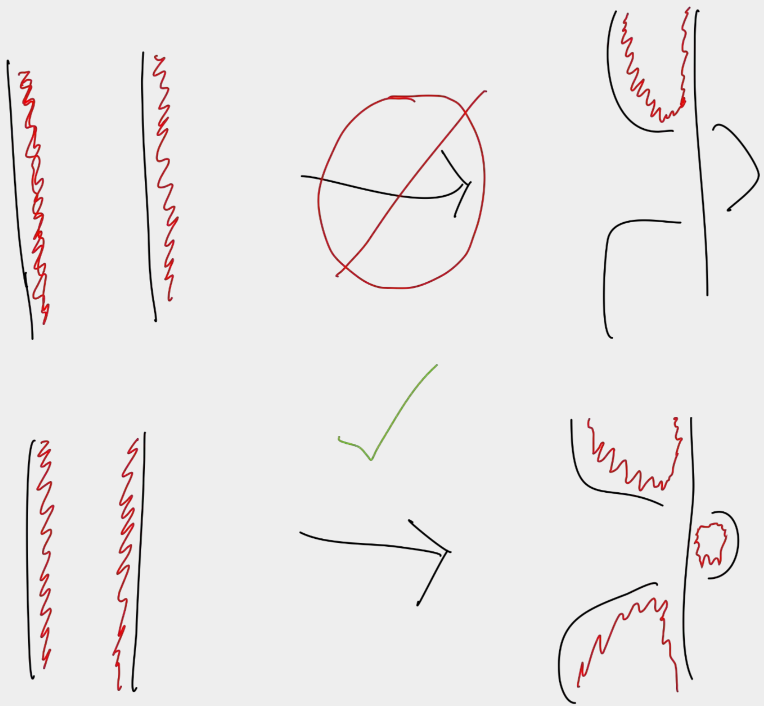
VR move Colorability



R_1 and R_3 moves



R_2 move Colorability



Why this is interesting?

Conjecture: Let $D_n(L)$ be the set of all diagrams of link L with number of virtual crossings, and number of classical crossings less than or equal to n. And let $C_n(L)$ be the subset of diagrams in $D_n(L)$ that are colorable, then $\lim_{n\to\infty} \frac{|C_n(L)|}{|D_n(L)|} = 1$ or 0.

Whenever it is possible to make an r_2 move it is also possible to make a vr_2 move. If r_2 move changes colorability at most $1/2$ of the time, then, we know that most moves preserve colorability.

Arrow Polynomial + Colorability

$$A^s(K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}).$$

Then the *k-degree* of this summand is defined to be

$$i_1 \times j_1 + i_2 \times j_2 + \cdots + i_v \times j_v,$$

Theorem 4.3. *Let D be an oriented checkerboard colorable virtual link diagram. Then*

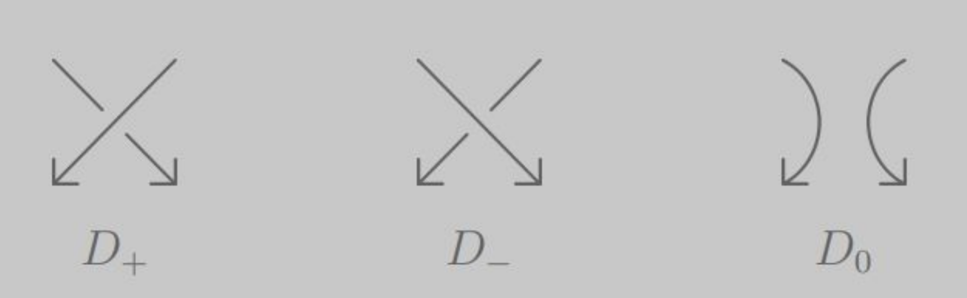
(1) $AS(D)$ only contains even integer; and

(2) for any summand $A^s K_{i_1}^{j_1} K_{i_2}^{j_2} \cdots K_{i_v}^{j_v}$ with $1 \le i_1 < i_2 < \cdots < i_v$, $j_t \ge 1$ for $t = 1, 2, \cdots, v$, and $v \ge 1$ of $\langle D \rangle_{NA}$, we have $2i_v \le \sum_{t=1}^{v} i_t \cdot j_t$. In particular, $\langle D \rangle_{NA}$ has no summands like $A^s K_i$.

Deng-Jin-Kauffman (2020)

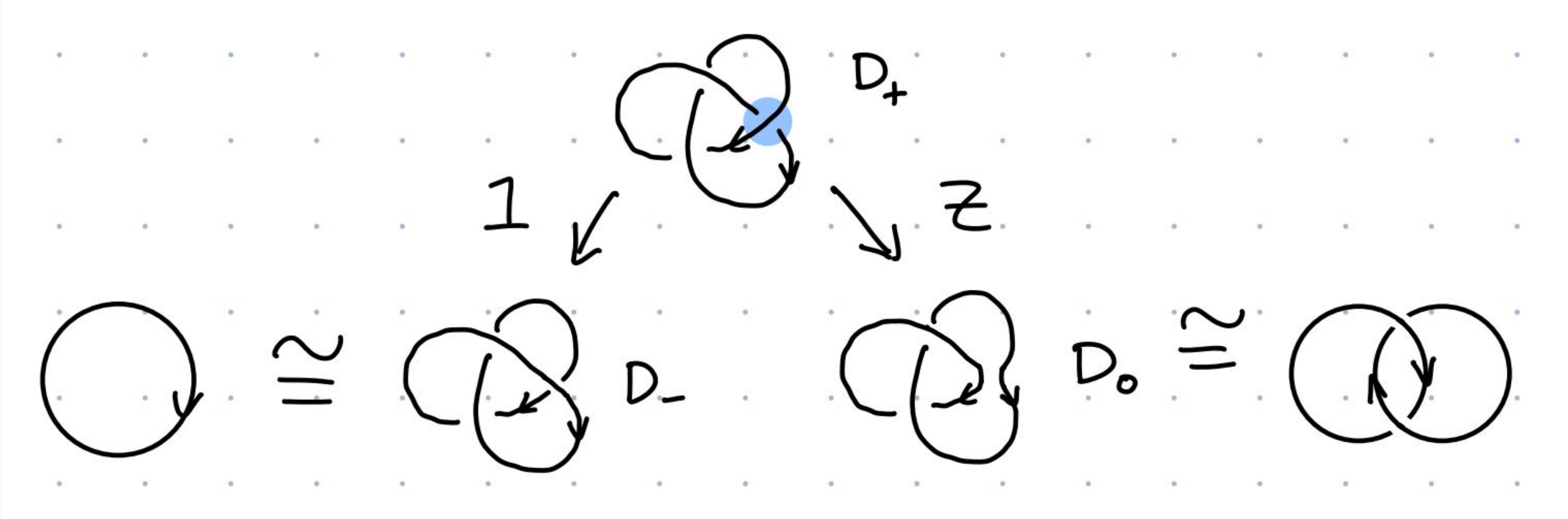
Alexander-Conway Polynomial

Definition: The Alexander-Conway Polynomial $\nabla_D(z) \in \mathbf{Z}[z]$ can be defined using the Skein relation $\nabla_{D+}(z) - \nabla_{D-}(z) = z \nabla_{D0}(z)$, where (D_+, D_-, D_0) is a Skein triple. That is, the diagrams for D_+, D_-, and D_0 are identical everywhere except a small region where the diagrams are as follows:

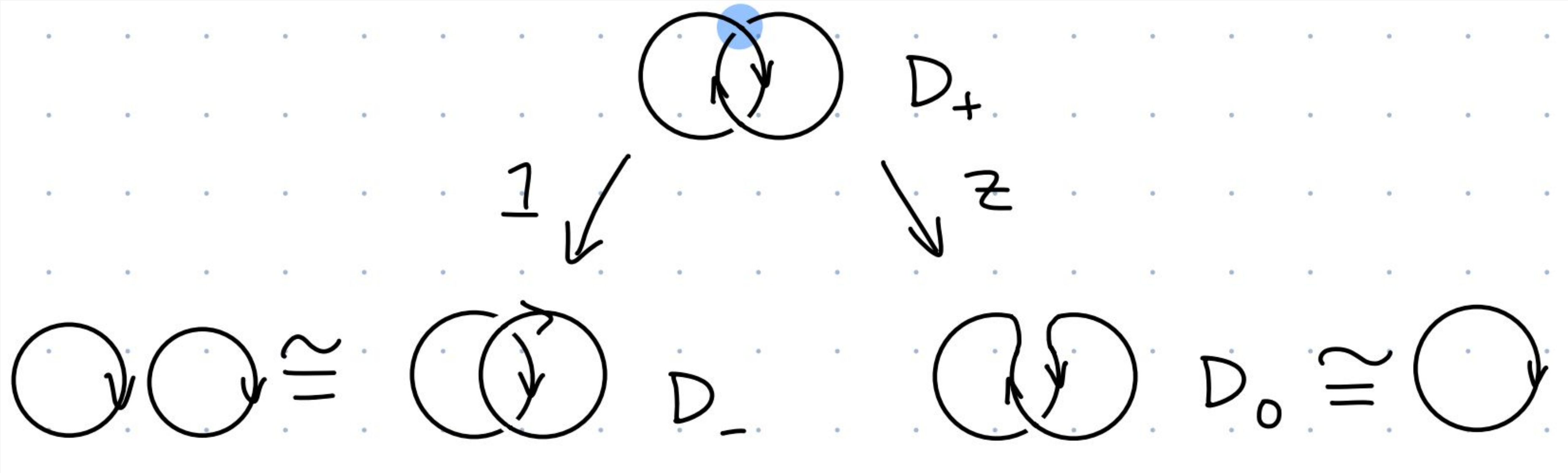


Remark: The Alexander-Conway polynomial is related to the Alexander polynomial $\Delta_L(z)$ (Alexander 1928) under the relation $\Delta_L(z - z^{-1}) = \nabla_L(z^2)$

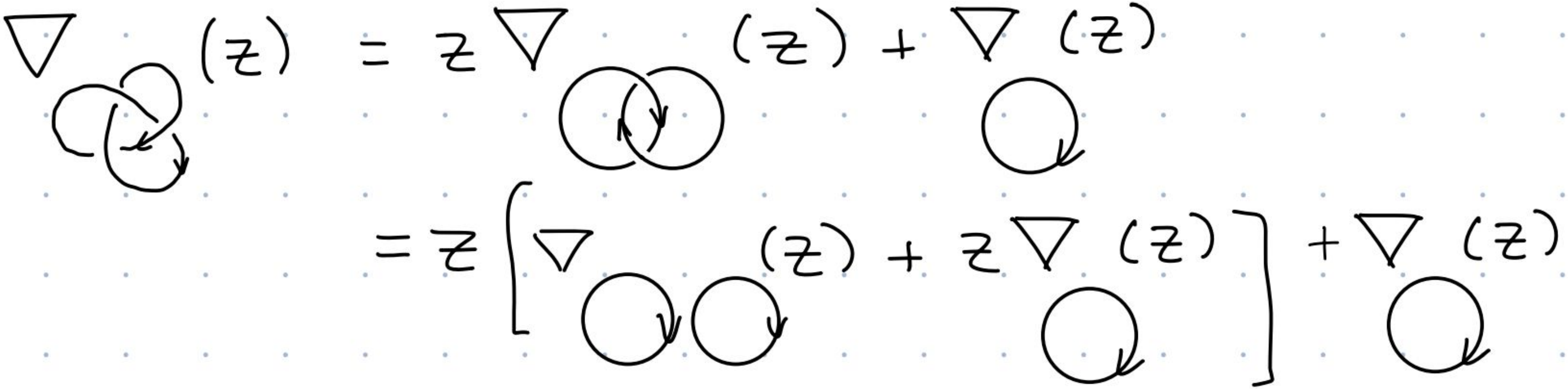
Alexander-Conway Polynomial Example



Alexander-Conway Polynomial Example cont.



Alexander-Conway Polynomial Example cont.



$$= z(0 + z) + 1$$

$$= z^2 + 1$$

Conway Polynomial

The Conway Polynomial is presented as formulated by J. Sawollek (Sawollek 1999).

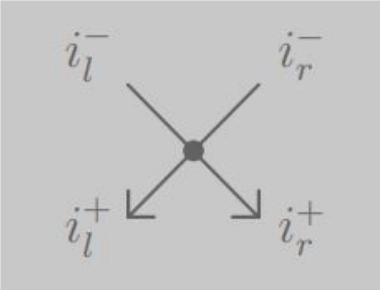
Definition: Let D be a virtual link diagram with $n \geq 1$ classical crossings $c_1, \ldots, c_n$. Define

$$M_+ := \begin{pmatrix} 1-x & -y \\ -xy^{-1} & 0 \end{pmatrix} \quad \text{and} \quad M_- := \begin{pmatrix} 0 & -x^{-1}y \\ -y^{-1} & 1-x^{-1} \end{pmatrix}.$$

For $i = 1, \ldots, n$, let $M_i := M_+$ if c_i is positive, and let $M_i := M_-$ otherwise. Define the $2n \times 2n$ matrix M as a block matrix by $M := \mathrm{diag}(M_1, \ldots, M_n)$.

Conway Polynomial cont.

Furthermore, consider the graph belonging to the virtual link diagram where the virtual crossings are ignored (i.e. the graph consists of n vertices $v_1, \ldots, v_n$ and 2n edges $e_1, \ldots, e_{2n}$). Subdivide the edges into two half edges and label them at each vertex v_i as follows:



A permutation of {1, …, n} x {l, r} is given by the assignment $(i, a) \mapsto (j, b)$ if the half edges i_a^+ and j_b^+ belong to the same edge of the virtual diagram's graph. Let P denote the corresponding 2n x 2n permutation matrix.

Conway Polynomial cont.

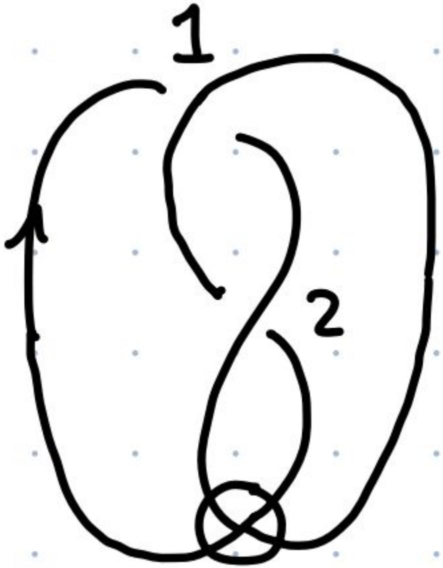
We can define $Z_D(x, y) := (-1)^{w(D)} \det(M - P)$, where $w(D)$ is the writhe of D. One can note that $Z_D(x, y)$ is an invariant of virtual links up to multiplication by a power of $x^{\pm 1}$.

One can define the normalized polynomial $Z'_D(x, y) := x^{-N} Z_D(x, y)$ where N is the lowest exponent in the variable x of the polynomial $Z_D(x, y)$. $Z'_D(x, y)$ is an invariant of virtual links.

Note that $Z_D(x, y)$ satisfies the skein relation

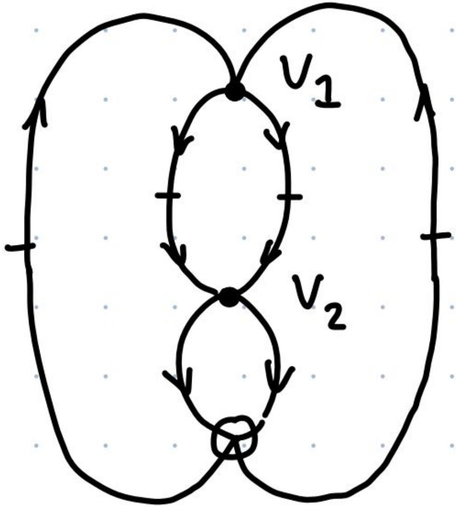
$$x^{-1/2} Z_{D+}(x, y) - x^{1/2} Z_{D-}(x, y) = (x^{-1/2} - x^{1/2}) Z_{D0}(x, y)$$

Example of Conway Polynomial



$$M = \begin{bmatrix} 1-x & -y & 0 & 0 \\ -xy^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1-x & -y \\ 0 & 0 & -xy^{-1} & 0 \end{bmatrix}$$

Example of Conway Polynomial (cont.)



$(1, \ell) \to (2, \ell)$

$(1, r) \to (2, r)$

$(2, \ell) \to (1, r)$

$(2, r) \to (1, \ell)$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} (1, \ell) \\ (1, r) \\ (2, \ell) \\ (2, r) \end{matrix}$$

Example of Conway Polynomial (cont.)

$$Z_0'(x,y) = Z(x,y) = (-1)^2 \det \begin{bmatrix} 1-x & -y & 0 & -1 \\ -xy^{-1} & 0 & -1 & 0 \\ -1 & 0 & 1-x & -y \\ 0 & -1 & -xy^{-1} & 0 \end{bmatrix}$$

$$= x^2 + x^2y^{-1} + xy - xy^{-1} - y - 1$$

A Result About Conway Polynomial

Let D, D_1, D_2 be virtual link diagrams and let $D_1 \sqcup D_2$ denote the disconnected sum of the diagrams D_1 and D_2. Then the following hold:

a) $Z_D(x, y) = Z'_D(x, y) = 0$ if D has no virtual crossings

b) $Z_{D1 \sqcup D2}(x, y) = Z_{D1}(x, y)\, Z_{D2}(x, y)$, and $Z'_{D1 \sqcup D2}(x, y) = Z'_{D1}(x, y)\, Z'_{D2}(x, y)$

Remark: For a connected sum $D_1 \# D_2$ of virtual link diagrams, a formula of the form

$$Z_{D1 \# D2}(x, y) = c\, Z_{D1}(x, y)\, Z_{D2}(x, y)$$

with a constant c does not hold in general.

Remark Regarding Connected Sums